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A mathematical model for the characteristic impedance: the dependence on temperature and frequency

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This paper is devoted to the estimation of the dependence of the characteristic impedance of the material on the temperature and the frequency of the incident electromagnetic field. The result is illustrated with plots of dependencies of moduli of real and imaginary parts of the characteristic impedance on the reduced temperature.

Key words: characteristic impedance, refractive index, superconductivity, electromagnetic field.

Робота присвячена оцінці залежності характеристичного імпедансу матеріалу від температури і частоти збуджуючого електромагнітного поля. Результат ілюстровано графіками залежності модулів дійсної та уявної частин характеристичного імпедансу від приведеної температури.

Ключові слова: характеристичний імпеданс, коефіцієнт заломлення, надпровідність, електромагнітне поле.

Работа посвящена оценке зависимости характеристического импеданса материала от температуры и частоты возбуждающего электромагнитного поля. Результат проиллюстрирован графиками зависимости модулей вещественной и мнимой частей характеристического импеданса от приведенной температуры.

Ключевые слова: характеристический импеданс, коэффициент преломления, сверхпроводимость, электромагнитное поле.

The rigorous set of the two-dimensional scattering and diffraction problem for a plane monochromatic electromagnetic wave which depends on time as $e^{-i\omega t}$ on non perfectly conducting objects leads to the Robin boundary condition for the Helmholtz equation as was shown in [1].

This boundary condition contains a parameter depending on the characteristic impedance of the material which the reflecting object consists of. We consider the materials whose atoms contain rigidly attached electrons and so-called free electrons providing the conduction of the material. The number of rigidly attached electrons is not considered. We consider a material in which all the electrons of the atoms are free and a material where the number of free electrons depends on the temperature. We have used the dependencies obtained in the papers of C. J. Gorter and H. Casimir [2], D. A. Bonn and coauthors [3], G. F. Dionne [4], O. G. Vendik, A. Y. Popov [5]. We are obtained the explicit formulae that enable us to estimate the characteristic impedance of materials. For all the dependencies ceteris paribus the plots of moduli of real and imaginary parts of the characteristic impedance were built and they are close to each other.

In this paper in contrast to other papers (e. g. [6-8]), the characteristic impedance is associated with the refraction coefficient of the electromagnetic wave. A physical interpretation of real and imaginary parts of this coefficient is given. For niobium and lead the plots of dependencies of moduli of real and imaginary parts of the characteristic impedance from the temperature were built. The dependencies of the real and imaginary parts of the impedance on the frequency were obtained.

The word “material” stands for a mathematical model of the material, the set of atoms. We assume that the atoms do not interact during the oscillation caused by exciting electromagnetic field. Thus to characterize the oscillation of the whole system it is sufficient to know the character of the oscillation of an atom.

The right orthogonal coordinate system is chosen so that the atom is situated at the point with zero coordinates and the z-axis is directed opposite to the wave vector of the exciting electromagnetic field; the x-axis direction coincides with the direction of the vector of the exciting electric field.

Consider an E modes electromagnetic wave falling on an atom. The electric field has the form $\overrightarrow{E^i}\left(x^i(t), y^i(t), z^i(t)\right) = \left(E_x^i(t), 0, 0\right)$, where $E_x^i(t) = E_x^i e^{-i\omega t}$. The magnetic field has the form $\overrightarrow{H^i}\left(x^i(t), y^i(t), z^i(t)\right) = \left(0, H_y^i(t), H_z^i(t)\right)$, where $H_y^i(t) = H_z^i e^{-i\omega t}$ and $H_z^i(t) = H_z^i e^{-i\omega t}$. We assume that forced oscillations of the electron do not depend on $x^i(t)$, $y^i(t)$, $z^i(t)$, y- and z-components of $\overrightarrow{H^i}\left(x^i(t), y^i(t), z^i(t)\right)$.

As in [9] we assume that the model of the atom of the material is a dissipative isotropic oscillator with own cyclic oscillation frequency ω_0, $\omega_0 = \sqrt{\frac{k}{m}}$, where k is a dependence coefficient linking the restoring force and the deviation of electron from the equilibrium position and m is the mass of the oscillating electron.

Under the effect of the electromagnetic field free electrons inside the atom begin forced oscillations. The isotropism provides the same restoring force of the electron for any direction of the electric field.

Thus three forces act on the electron. The first is an outside force caused by the electric field. The second is a dissipative force equal to $-\gamma \dot{x}(t)$ where γ is the dissipation coefficient and $x(t)$ is a coordinate. The third is the restoring force equal to $-\omega_0^2 x(t)$.

Let us denote the force acting on the electron from the side of the field as F_x then $F_x = q_e E_x^i e^{-i\omega t}$ where q_e is the electron charge. Then the second Newton law will take the following form

$$m\ddot{x}(t) = q_e E_x^i e^{-i\omega t} - m\gamma \dot{x}(t) - m\omega_0^2 x(t). \tag{1}$$

We are going to search the solution of (1) in the form $x(t) = x_0 e^{-i\omega t}$ where x_0 is a constant. We get that

$$x(t) = \frac{q_e}{m\left(-\omega^2 - i\omega\gamma + m\omega_0^2\right)} E_x^i(t). \tag{2}$$

From (2) one can see that $x(t)$ is proportional to $E_x^i(t)$. Similarly, we can prove that in the case of exciting of the material by H modes electromagnetic wave $x(t)$ is proportional to $H_x^i(t)$.

The projection of the induced dipole moment of the atom on the x-axis is given in [9] by the formula $p_x = \varepsilon_0 \alpha(\omega) E_x^i(t)$ where $\alpha(\omega)$ is the atomic polarizability. From the other hand $p_x = q_e x(t)$ we have

$$p_x = \varepsilon_0 \frac{q_e^2}{m\varepsilon_0\left(-\omega^2 - i\omega\gamma + m\omega_0^2\right)} E_x^i(t)$$

therefore $\alpha(\omega) = \frac{q_e^2}{m\varepsilon_0\left(-\omega^2 - i\omega\gamma + m\omega_0^2\right)}$.

Let us denote the polarization vector by $\vec{P}$ and the number of free electrons in the atom by N. Then the dependence of the projection of $\vec{P}$ on x-axis on N will take the form $P_x = \varepsilon_0 N \alpha(\omega) E_x$ (see [9]).

Consider a material that fill the half of the space that is the set $\left\{(x, y, z) \in \mathbb{R}^3 : x \in \mathbb{R}, y \in \mathbb{R}, z \le 0\right\}$ consisting of atoms whose model is presented above. The exciting field is the same. The Maxwell equations in the matter have the following form

$$\begin{cases} rot\overrightarrow{E^e}(z,t) = -\dfrac{\partial \overrightarrow{B^e}(z,t)}{\partial t}, \\ rot\overrightarrow{H^e}(z,t) = -\dfrac{\partial \overrightarrow{D^e}(z,t)}{\partial t}, \\ div\overrightarrow{E^e}(z,t) = -\dfrac{1}{\varepsilon_0} div\vec{P}, \\ div\overrightarrow{H^e}(z,t) = 0, \end{cases}$$

where $\overrightarrow{E^e}$ is the electric field, $\overrightarrow{H^e}$ is the magnetic field, $\overrightarrow{B^e}$ is the magnetic induction and $\overrightarrow{D^e}$ is the electric induction.

Knowing that $\overrightarrow{D^e} = \varepsilon_0 \overrightarrow{E^e} + \vec{P}$ we get

$$\frac{\partial^2 E_x^e(z,t)}{\partial z^2} = \frac{1}{c^2}\left(1 + N\alpha(\omega)\right)\frac{\partial E_x^e(z,t)}{\partial t^2}. \qquad (3)$$

The length of the electromagnetic wave in the matter is defined by the formula $\lambda = \frac{2\pi}{\omega} \cdot \frac{c}{n}$ where n is the refraction coefficient. The change of the wave length is the result of the imposition and the interference of the incident wave and the waves caused by the oscillating electrons.

We are going to search the solution of (3) in the form $E_x^e(z,t) = E_x^e e^{-i\omega t} e^{ikz}$ where

$k = \frac{2\pi}{\lambda}$ is the wave number. We get that

$$k^2 = \omega^2 \frac{1}{c}\left(1 + N\alpha(\omega)\right). \tag{4}$$

From (4) using the formula $k = \frac{\omega n}{c}$ we get

$$n^2 = 1 + N\alpha(\omega) = 1 + \frac{Nq_e^2}{m\varepsilon_0} \frac{1}{-\omega^2 - i\omega\gamma + m\omega_0^2}. \tag{5}$$

The number $\sqrt{\frac{Nq_e^2}{m\varepsilon_0}}$ is called the plasma frequency and is denoted by ω_p. For radio waves up to ultra short waves the condition of smallness of ω compared to ω_p takes place. For free electrons ω_0 is equal to zero.

For materials that model ordinary elements γ is not equal to zero and for materials that model superconducting elements γ is equal to zero. Consider ordinary elements. Ignoring 1 and ω^2 in (5) we get

$$n^2 = i\frac{Nq_e^2}{m\varepsilon_0}\frac{1}{\omega\gamma}.$$

In the equation whose solution describes the motion of the electron, the force resisting its motion $-m\gamma\dot{x}(t)$ was used. As the averaged motion of electrons under the effect of the electric field is uniform (see [9]) the averaging of the resistance force must be equal to the force acting on an electron from the side of the electric field. We obtain that $m\gamma v = q_e E_x$ where v is the averaged velocity. From the other hand we have $v = \frac{q_e E_x}{m} \cdot \tau$ where τ is the average time of the free path of the electron. Therefore $\tau = \gamma^{-1}$.

The conductivity of a material is given by formula $\sigma = \frac{Nq_e^2}{m} \cdot \tau$ then the refraction coefficient will take the form $n^2 = \frac{i\sigma}{\varepsilon_0 \omega}$. Using the following fact

$$\sqrt{a + ib} = \pm\left(\sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}} + i\,\mathrm{sgn}\, b\sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}\right)$$

finally we obtain the formula for the refraction coefficient

$$n = (1 + i)\sqrt{\frac{Nq_e^2\tau}{2\varepsilon_0\omega m}}, \tag{6}$$

where the branch of the square root is chosen such that $\operatorname{Im} n \geq 0$. This condition provides the decrease of electromagnetic wave amplitude during the penetration of the wave into the material. Substituting (6) in $E_x^e(z,t) = E_x^e e^{-i\omega t} e^{ikz}$ we obtain

$$E_x^e(z,t) = E_x^e e^{-i\omega t} e^{iz\sqrt{\frac{\omega N q_e^2 \tau \mu_0}{2m}}} e^{-\frac{z}{\sqrt{\frac{2m}{\omega N q_e^2 \tau \mu_0}}}}.$$

Thus one can see that the real part of (6) defines the change of the wave length in the material and the imaginary part defines the coefficient of wave attenuation.

Now we consider an equation $rot\overrightarrow{H^e}(z,t) = -\frac{\partial}{\partial t}\overrightarrow{D^e}(z,t)$. We have that

$$rot\overrightarrow{H^e}(z,t) = -\varepsilon_0 n^2 \frac{\partial \overrightarrow{E^e}(z,t)}{\partial t}. \quad (7)$$

In E-mode case the equation (7) will take the following form

$$\frac{\partial H_y^e(z,t)}{\partial z} = -\varepsilon_0 n^2 \frac{\partial E_y^e(z,t)}{\partial t}.$$

Let $H_y^e(z,t) = H_y^e e^{-i\omega t} e^{ikz}$ then

$$E_x^e = \frac{1}{c\varepsilon_0 n} H_y^e.$$

Denote the characteristic impedance by Z. Finally we obtain that

$$Z = (1-i)\sqrt{\frac{m\mu_0\omega}{2Nq_e^2\tau}}.$$

Consider a material all the electrons of which form a superconducting system i.e. all of them are free. In this case ω_0 and γ are equal to zero. Ignoring 1 in (5) we get

$$n^2 = -\frac{Nq_e^2}{m\varepsilon_0}\frac{1}{\omega^2},$$

$$n = i\sqrt{\frac{Nq_e^2}{m\varepsilon_0\omega^2}}.$$

Thus the refraction coefficient is pure imaginary.

Using arguments similar to presented above we obtain a formula for the characteristic impedance

$$Z = -i\sqrt{\frac{m\mu_0\omega^2}{Nq_e^2}}.$$

Now we consider a material in which free electrons form two subsystems a normal one with particles number density N_n and a superconducting one with particles number density N_s. Note that $N = N_n + N_s$. Each of them gives a contribution to polarizability of the material with atomic polarizabilities $\alpha_n(\omega)$ and $\alpha_s(\omega)$ respectively. From the Maxwell equations we have

$$\frac{\partial^2 E_x^e(z,t)}{\partial z^2} = \frac{1}{c^2}\left(1 + N_n\alpha_n(\omega) + N_s\alpha_s(\omega)\right)\frac{\partial E_x^e(z,t)}{\partial t^2}.$$

It follows that for the square of the refraction coefficient we have

$$n^2 = 1 + \frac{N_n q_e^2}{m\varepsilon_0} \frac{1}{-\omega^2 - i\omega\gamma + m\omega_0^2} + \frac{N_s q_e^2}{m\varepsilon_0} \frac{1}{-\omega^2 - i\omega\gamma + m\omega_0^2}. \quad (8)$$

Ignoring the first term in (8) and $-\omega^2 + m\omega_0^2$ in the denominator of the second term and noticing that in the third term ω_0 and γ are equal to zero we obtain

$$n^2 = -\frac{N_n q_e^2}{m\varepsilon_0} \frac{1}{i\omega\gamma} - \frac{N_s q_e^2}{m\varepsilon_0} \frac{1}{\omega^2} = \left(\frac{\omega_p}{\omega}\right)^2 \left(-\frac{N_s}{N} + i\frac{N_n}{N}\omega\tau\right).$$

Taking the square root we get

$$n = \frac{\omega_p}{\omega\sqrt{2}} \sqrt{\sqrt{\left(\frac{N_s}{N}\right)^2 + \left(\frac{N_n}{N}\omega\tau\right)^2} - \frac{N_s}{N}} + i\frac{\omega_p}{\omega\sqrt{2}} \sqrt{\sqrt{\left(\frac{N_s}{N}\right)^2 + \left(\frac{N_n}{N}\omega\tau\right)^2} + \frac{N_s}{N}}$$

where the branch of the square root is chosen such that $\operatorname{Im} n \geq 0$. This condition provides the decrease of electromagnetic wave amplitude during the penetration of the wave into the material. The characterization impedance is obtained from the following formula

$$Z = \frac{1}{c\varepsilon_0 n}.$$

Thus the formula for the characteristic impedance will take the following form

$$Z = \frac{\omega\sqrt{\sqrt{\left(\frac{N_s}{N}\right)^2 + \left(\frac{N_n}{N}\omega\tau\right)^2} - \frac{N_s}{N}}}{c\varepsilon_0\omega_p\sqrt{2\left(\left(\frac{N_s}{N}\right)^2 + \left(\frac{N_n}{N}\omega\tau\right)^2\right)}} - i\frac{\omega\sqrt{\sqrt{\left(\frac{N_s}{N}\right)^2 + \left(\frac{N_n}{N}\omega\tau\right)^2} + \frac{N_s}{N}}}{c\varepsilon_0\omega_p\sqrt{2\left(\left(\frac{N_s}{N}\right)^2 + \left(\frac{N_n}{N}\omega\tau\right)^2\right)}}.$$

Consider a function from the reduced temperature $f(t_n) = \frac{N_n}{N}$ where $t_n = \frac{T}{T_n}$ and T_n is a temperature of the superconducting transition of the material. Note that $\frac{N_s}{N} = 1 - f(t_n)$. Then using that $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$ and $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ we obtain the following form of the formula for the characteristic impedance

$$Z = \frac{\omega Z_0}{\sqrt{2}\omega_p}\left(\sqrt{\frac{\sqrt{(1 - f(t_n))^2 + (f(t_n)\omega\tau)^2} - 1 + f(t_n)}{(1 - f(t_n))^2 + (f(t_n)\omega\tau)^2}} - \right.$$

$$\left. -i\sqrt{\frac{\sqrt{(1 - f(t_n))^2 + (f(t_n)\omega\tau)^2} + 1 - f(t_n)}{(1 - f(t_n))^2 + (f(t_n)\omega\tau)^2}}\right).$$

The average time of free path of a electron can be estimated using the formula

$\tau = \frac{m\sigma}{N_n q_e^2}$ taking into account the known minimal value of the conduction at a temperature over T_n. The simplest formula for $f(t_n)$ is given in the model of C. J. Gorter and H. Casimir [2]. It is $f(t_n) = t_n^4$. There are other models. The model of G. F. Dionne [4] is $f(t_n) = e^{W - \frac{W}{t_n}}$, the model of D. A. Bonn [3] is $f(t_n) = t_n^{4-t_n}$, the model of O. G. Vendik and A. Y. Popov [5] is $f(t_n) = t_n^{\frac{3}{2}}$.

To calculate the dependence of the characteristic impedance on the temperature we used the following reference information:

− the absolute zero of the temperature is equal to $-273{,}15$ degrees Celsius;

− the boiling point of helium is equal to $4{,}21$ degrees Kelvin;

− the boiling point of hydrogen is equal to $20{,}4$ degrees Kelvin.

For niobium:

− the temperature of the superconducting transition is equal to $9{,}22$ degrees Kelvin;

− the ratio of specific conductivities at $20{,}3$ degrees Kelvin and 273 degrees Kelvin is equal to $0{,}338$;

− the critical magnetic field strength is equal to $318{,}32 \cdot 10^3$ A/m;

− the molar mass is equal to $41 \cdot 10^{-3}$ kg/mol;

− the density is equal to $8{,}4 \cdot 10^3$ kg/m^3;

− the electron density is equal to $6{,}17 \cdot 10^{29}$ 1/m^3;

− the plasma frequency is equal to $4{,}43 \cdot 10^{16}$ Hz;

− the specific resistance at 273 degrees Kelvin is equal to $13{,}1 \cdot 10^{-8}$ Om·m;

− the time of free path of an electron is equal to $1{,}30 \cdot 10^{-15}$ s;

− the frequency of exciting field is taken equal to 30 GHz and the associated wave length is equal to 1 sm.

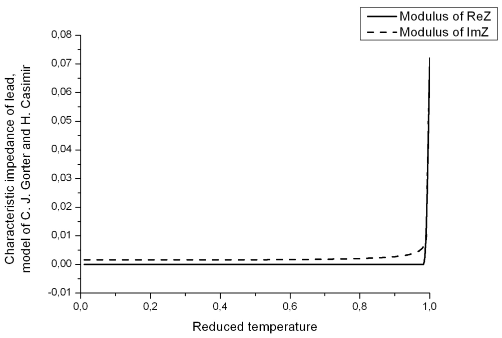

Fig. 1.

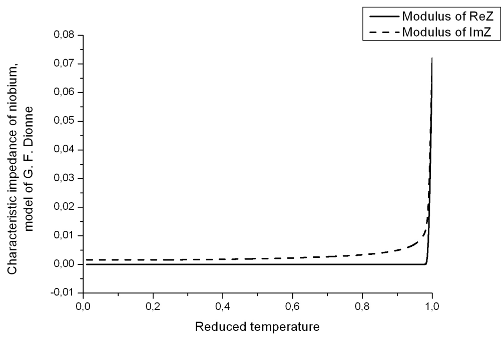

Fig. 2.

On the figure 1 the dependence of moduli of real and imaginary parts of the characteristic impedance on the reduced temperature under the assumption that $f\left(t_n\right)=t_n^4$ is shown. On the figure 2 the same is shown under the assumption that $f\left(t_n\right)=e^{W-\frac{W}{t_n}}$ where $W=1$. On the figure 3 we assume that $f\left(t_n\right)=t_n^{4-t_n}$ and on the figure 4 we assume that $f\left(t_n\right)=t_n^{\frac{3}{2}}$. The solid line stands for modulus of the real part and the dash line is for modulus of the imaginary part.

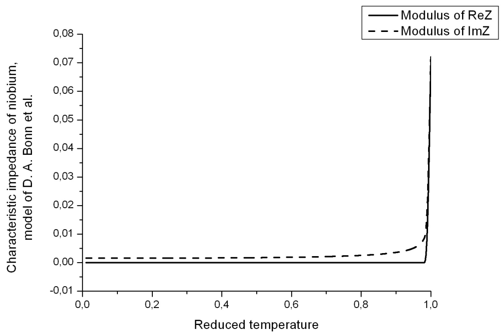

Fig. 3.

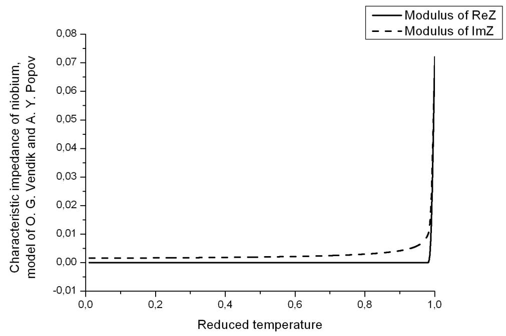

Fig. 4.

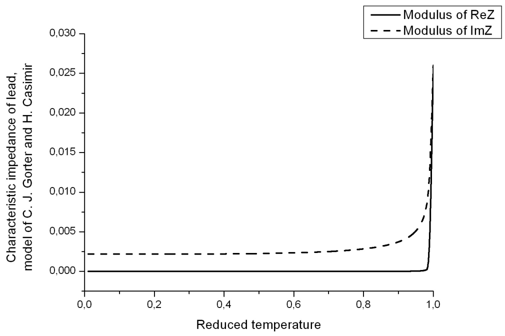

Fig. 5.

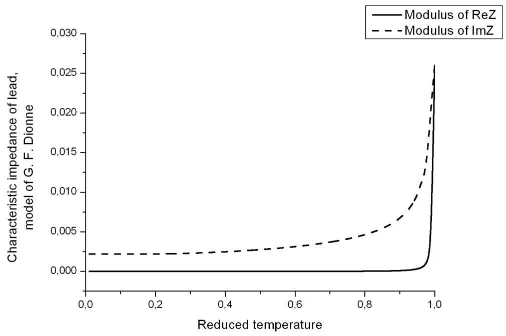

Fig. 6.

For lead:

− the temperature of the superconducting transition is equal to $7,26$ degrees Kelvin;

− the ratio of specific conductivities at $20,5$ degrees Kelvin and 273 degrees Kelvin is equal to $0,0301$;

− the critical magnetic field strength is equal to $63,664\cdot 10^3$ A/m;

− the molar mass is equal to $82\cdot 10^{-3}$ kg/mol;

− the density is equal to $11,34\cdot 10^3$ kg/m^3;

− the electron density is equal to $3,33\cdot 10^{29}$ 1/m^3;

− the plasma frequency is equal to $3,25\cdot 10^{16}$ Hz;

− the specific resistance at 273 degrees Kelvin is equal to $19,3\cdot 10^{-8}$ Om·m;

− the time of free path of an electron is equal to $1,83\cdot 10^{-14}$ s;

− the frequency of exciting field is taken equal to 30 GHz and the associated wave length is equal to 1 sm.

On the figure 5 the dependence of moduli of real and imaginary parts of the characteristic impedance on the reduced temperature under the assumption that $f(t_n)=t_n^4$ is shown. On the figure 6 the same is shown under the assumption that $f(t_n)=e^{W-\frac{W}{t_n}}$ where $W=1$. On the figure 7 we assume that $f(t_n)=t_n^{4-t_n}$ and on the figure 8 we assume that $f(t_n)=t_n^{\frac{3}{2}}$. The solid line stands for modulus of the real part and the dash line is for modulus of the imaginary part.

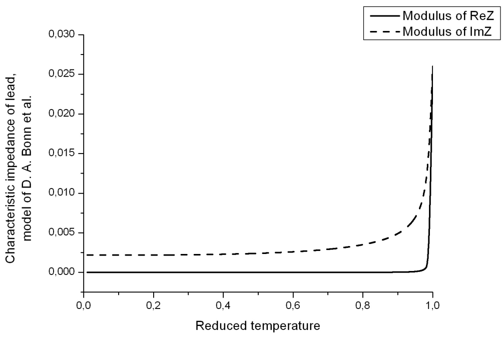

Fig. 7.

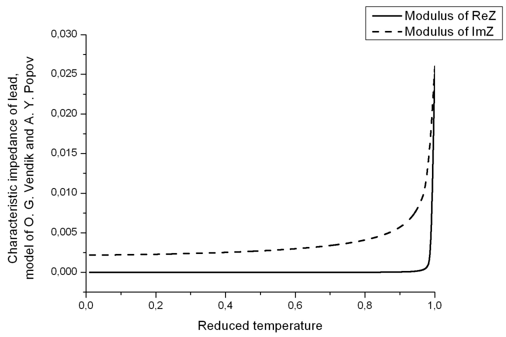

Fig. 8.

The formulae obtained to calculate the characteristic impedance enable us to estimate its value for different materials at different temperatures and frequencies. For materials in normal state the real part and the imaginary part of the characteristic impedance are equal and are proportional to the square root of the frequency. One can easily see that real part of the characteristic impedance of a material modeling a metal in the superconducting state is proportional to the square of the frequency and imaginary part is proportional to the first degree. Besides, the imaginary part of the characteristic impedance is greater than the real part by four to five orders of magnitude for all values of the reduced temperature except for those close to unity.

The obtained formulae enabling us to estimate the characteristic impedance of different materials let us get physically based solutions of wave scattering and diffraction problems. The method of obtaining such solutions was proposed in [1] developed in [10] widely presented in [11] and modified in [12].

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