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On representation of a probabilistic finite-state automaton as a composition of a Markov chain and a deterministic finite-state automaton

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Probabilistic finite-state machines such as probabilistic finite-state automata, Markov chains and probabilistic suffix trees are used today in a wide amount of fields in pattern recognition, or in fields to which pattern recognition is linked: computational linguistics, bioinformatics and machine translation. In the present paper we formulate a criterion for determining when a probabilistic finite-state automaton can be represented as a composition of a Markov chain and a deterministic finite-state automaton.

Introduction

Probabilistic finite-state machines such as probabilistic finite-state machines such as probabilistic finite-state automata, hidden Markov models, Markov chains, probabilistic suffix trees are used today in a wide amount of fields in pattern recognition, or in fields to which pattern recognition is linked: computational linguistics, bioinformatics and machine translation [1-4].

One of the most interesting and perspective research objects is a probabilistic finite-state automaton (PFA). The characteristics of a finite-state deterministic automaton (DFA) and a Markov chain (MC) are quite well studied. That's why it makes sense to try to reduce the investigation of a PFA to the investigation of the behavior of these machines.

In the present paper we formulate a criterion for determining when a PFA can be represented as a composition of a MC and a DFA.

Initial concepts

Let's introduce a few definitions.

Definition 1

Probabilistic finite-state automaton (PFA) is a 5-tuple

$$M=\left(Q_M,\Sigma_M,P,q_{0M},F_M\right) \qquad (1)$$

Q_M – a finite set of states;

Σ_M – a finite alphabet;

P – a mapping defining the transition probability function

$P:Q_M\times\Sigma_M\times Q_M\to \mathrm{R}^+$

$\mathrm{R}^+=\{x\in \mathrm{R}\mid x\geq 0\}$

$$\left(\forall(q',a)\in Q\times\Sigma\right)\left(\sum_{q''\in Q}P(q',a,q'')=1\right)$$

q_{0M} – an initial state;

F_M – a set of admissive states.

Definition 2

Deterministic finite-state automaton (DFA) is a 5-tuple

$$D=(Q_D,\Sigma_D,T,q_{0D},F_D) \tag{2}$$

Q_D – a finite set of states;

Σ_D – a finite alphabet;

T – a mapping defining the transition function. For convenience we may consider that T represents a transition graph between states.

$T:Q_D\times\Sigma_D\to Q_D$

q_{0D} – an initial state,

F_D – a set of admissive states.

So the main difference between a DFA and a PFA is their transition function.

Definition 3

Markov chain (MC) p is defined by a transition matrix:

$$\mathrm{p}=\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m}\\ p_{21} & p_{22} & \cdots & p_{2m}\\ \vdots & & & \vdots\\ p_{m1} & p_{m2} & \cdots & p_{mm}\end{pmatrix} \tag{3}$$

where p_{ij} is a probability of a transition from state q_i to state q_j

We may notice that defining matrix p means defining a function

$\mathrm{p}:Q_{\mathrm{p}}\times Q_{\mathrm{p}}\to \mathrm{R}^+$, $\mathrm{p}(q_i,q_j)=p_{ij}$

Q_{p} – a finite set of states.

$$\left(\forall q'\in Q_\pi\right)\left(\sum_{q''\in Q_\pi}\pi(q',q'')=1\right)$$

Definition 4

Let

$(q_1,a_1)\approx(q_2,a_2)$, when $(\forall q\in Q)\left(P(q_1,a_1,q)=P(q_2,a_2,q)\right)$.

This relation is an equivalence relation. Hence, it breaks set $U = \{(q,a) \in Q \times \Sigma\}$ into equivalence classes $U = \bigcup_{i=1}^{i=k} U_i$ where

$U_i = \{(q_i, a_i) \in Q \times \Sigma \mid (\forall \{(q', a'), (q'', a'')\} \subset U_i)((q', a') \approx (q'', a''))\}$

Definition 5

In order to consider a composition of automata we need that $(\Sigma_M = \Sigma_D = \Sigma) \wedge (Q_M = Q_D = Q_\pi = Q) \wedge (F_M = F_D = F)$.

PFA (1) can be represented as a composition of DFA (2) and a MC (3) if such functions T and π exist that $(\forall (q', a, q'') \in Q \times \Sigma \times Q)(P(q', a, q'') = \pi(T(q', a), q''))$.

Main part

Lemma

If PFA M is a composition of MC π and DFA D, then for T the following relationship holds true:

$$(\forall \{(q_1, a_1), (q_2, a_2) \subset Q \times \Sigma, (q_1, a_1) \neq (q_2, a_2)\})$$
$$(T(q_1, a_1) = T(q_2, a_2) \Rightarrow (q_1, a_1) \approx (q_2, a_2)).$$

□

Assume the contrary:

$$(\exists \{(q_1, a_1), (q_2, a_2) \subset Q \times \Sigma, (q_1, a_1) \neq (q_2, a_2)\})$$
$$(T(q_1, a_1) = T(q_2, a_2) \wedge \overline{(q_1, a_1) \approx (q_2, a_2)})$$

Let $T(q_1, a_1) = T(q_2, a_2) = q'$.

$$\overline{(q_1, a_1) \approx (q_2, a_2)} \Leftrightarrow (\exists q'' \in Q)(P(q_1, a_1, q'') \neq P(q_2, a_2, q'')) \quad (*)$$

$$P' = P(q_1, a_1, q'') = \pi(T(q_1, a_1), q'') = \pi(q', q'')$$
$$P'' = P(q_2, a_2, q'') = \pi(T(q_2, a_2), q'') = \pi(q', q'')$$

We obtain that $P' = P''$. However, according to (*) $P' \neq P''$.

We have a contradiction.

■

Let's consider the following bipartite graph $G = \langle V, E \rangle$: (4)

$X = U$

$Y = Q$

$V = X \cup Y$

$E = \{(x, y) \in X \times Y\}$

Theorem (criterion for determining when a PFA can be represented as a composition of a MC and a DFA)

A PFA (1) may be represented as a composition of a DFA (2) and a MC (3) if and only if a matching of the graph (4) exists which contains $|X|$ edges.

□

1. (Necessity)

By hypothesis PFA M is represented as a composition of DFA D and MC π.

Let $W = \{(U_i, T(q,a)) \mid U_i \in U, (q,a) \in U_i\}$.

Then W is a desired matching,

since

1) $|W| = |X|$;

2) $(\forall q \in Q)((\exists U_i \in U : (U_i, q) \in W) \Rightarrow (\exists! U_i \in U \mid (U_i, q) \in W))$.

Let us assume the contrary.

Then $(\exists q \in Q)(\exists U_i \in U \wedge \exists U_j \in U \mid U_i \neq U_j, (U_i, q) \in W, (U_j, q) \in W)$.

From the definition of W we may conclude that

$(\exists (q', a') \in U_i, \exists (q'', a'') \in U_j)(T(q', a') = q \wedge T(q'', a'') = q)$.

Using Lemma we obtain that $(q', a') \approx (q'', a'')$ but $U_i \neq U_j$.

We reach a contradiction.

2. (Sufficiency)

Let $W = \{(U_i, q_i')\}_{i=1}^{i=|X|}$ be a matching which consists of $|X|$ edges.

$U_i = \{(q_i, a_i) \in Q \times \Sigma \mid (\forall \{(q', a'), (q'', a'')\} \subset U_i)((q', a') \approx (q'', a''))\}$

Let $(\forall i \in [1, |X|])(\forall (q,a) \in U_i)(T(q,a) = q')$,

$$\pi(q', q'') = \begin{cases} \pi(T(q_i, a_i), q'') = P(q_i, a_i, q'') if (\exists (U_i, q') \in W, (q_i, a_i) \in U_i) \\ 0 \; otherwise \end{cases}$$

Therefore, from the construction follows $P(q', a, q'') = \pi(T(q', a), q'')$.

■

Corollary

A PFA (1) may be represented as a composition of a DFA (2) and a MC (3) if and only if $|U| \le |Q|$.

□

It is obvious that the matching of the graph (4) which consists of $|X|$ edges exists if and only if $|U| \le |Q|$.

■

It is also interesting to consider a question: how many ways are there to represent a PFA as a composition of a DFA and a MC?

Theorem

If a PFA (1) may be represented as a composition of a DFA (2) and a MC (3) then there are $A_{|Q|}^{|U|}$ ways to do this.

□

From the construction of the graph (4) one can easily see that there are $A_{|Q|}^{|U|}$ matchings.

■

Now examine an example.

Let us consider a PFA shown in Fig. 1. In this case $|U| = 4$, $|Q| = 5$. So $|Q| > |U|$ and we may conclude that this PFA can be represented as a composition of a DFA and a MC.

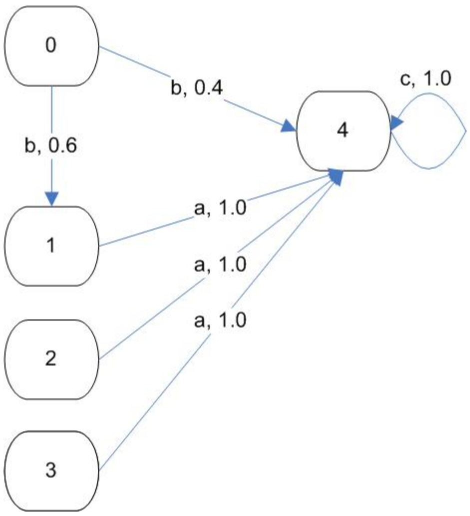

Fig. 1 Example of a PFA

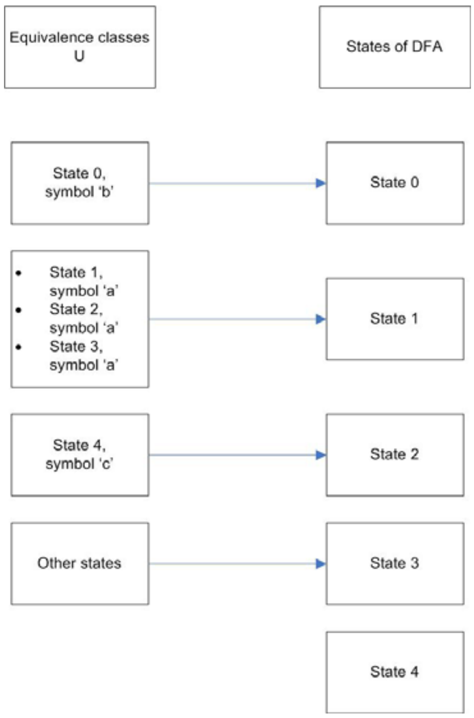

Fig. 2 Correspondence between equivalence classes U and states of a DFA

Let us find T and π. In order to do this we need to set up a correspondence between equivalence classes U and states of a DFA. The example of such a correspondence is shown in Fig. 2.

Table 1. Matrix T

State \ Input symbol	a	b	c
0	3	0	3
1	1	3	3
2	1	3	3
3	1	3	3
4	3	3	2

Table 2. Matrix π

State \ State	0	1	2	3	4
0	0	0	0	0	0.4
1	0	0	0	0	1.0
2	0	0	0	0	1.0
3	0	0.6	0	0	0
4	0	0	0	0	0

Using a few examples let us examine if the found values of functions T and π satisfy definition 5:

$$P(q_0,b,q_4)=P(T(q_0,b),q_4)=P(q_0,q_4)=0.4$$

$$P(q_0,b,q_1)=P(T(q_0,b),q_1)=P(q_3,q_1)=0.6$$

$$P(q_1,a,q_4)=P(T(q_1,a),q_4)=P(q_1,q_4)=1.0$$

Summary

The theorem proved in this paper gives a desired criterion for determining when a PFA can be represented as a composition of a MC and a DFA

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