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Boundary Integral Equations in Dynamic Contact Problems for Plates

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The existence of distributional solutions is investigated for the time-dependent bending of thin elastic plates with transverse shear deformation in the dynamic contact (transmission) problem. The dynamic analogues of the single and double layer potentials are introduced and their properties are studied. Four representations for the solutions to the contact problem in terms of these potentials are considered. These representations lead to four systems of boundary integral equations with respect to the unknown densities of the potentials. The existence of weak solutions to these systems is proved. Mathematics Subject Classification (2000). 74H20, 74K20,35L55,45F15.

1. Formulation of the problem

Elastic plates form a very important class of mechanical structures, so their rigorous mathematical analysis is an attractive and useful subject of study. Boundary value problems for the equilibrium equations for plates with transverse shear deformation (TSD) have already been solved by the authors by means of potential methods in terms of classical or weak solutions (see [1]-[3].) The basic dynamic problems are investigated in [4]. Below we restrict our attention to a dynamic contact problem for an infinite plate with a finite inclusion.

Consider a homogeneous and isotropic elastic plate of thickness h_0 which occupies a region $\overline{S}\times[-h_0/2,h_0/2]$, where S is a domain in R^2 with boundary ∂S. In the TSD model proposed in [1] it is assumed that the displacement vector at (x,x_3), $x\in R^2$, at time $t\geq 0$ is of the form $(x_3u_1(x,t),x_3u_2(x,t),u_3(x,t))$. Then $u=(u_1,u_2,u_3)$ satisfies the equation of motion

$$B(\partial_t^2 u)(x,t)+Au(x,t)=q(x,t), \qquad (x,t)\in G=S\times(0,\infty); \tag{1}$$

here $B=diag\{\rho h^2,\rho h^2,\rho\}$, $h^2=h_0^2/12$, ρ is the constant density of the material and A is a 3-by-3 second order matrix differential operator whose elements are expressed in terms of the Lame constants λ,μ of the material. Along with A we consider the boundary moment-force differential operator T. The explicit form of A and T may be found in [1]-[4].

Suppose that ∂S is a simple closed C^2-curve that divides R^2 into interior and exterior domains S^+ and S^-. We write $G^\pm=S^\pm\times(0,\infty)$, $\Gamma=\partial S\times(0,\infty)$ and assume that the regions S^+ and S^- are occupied by plates with different Lame constants, densities and thickness parameters. These parameters give rise to distinct matrices $B_\pm$ and matrix differential operators $A_\pm$ and $T_\pm$ corresponding to $S^\pm$, respectively.

The classical contact problem consists in finding $u_\pm \in C^2(G^\pm)\bigcap C^1(\overline{G}^\pm)$ satisfying

$$\begin{aligned}
&B_+(\partial_t^2 u_+)(x,t)+(A_+u_+)(x,t)=0, && (x,t)\in G^+,\\
&B_-(\partial_t^2 u_-)(x,t)+(A_-u_-)(x,t)=0, && (x,t)\in G^-,\\
&u_+(x,0+)=(\partial_t u_+)(x,0+)=0, && (x,t)\in S^+,\\
&u_-(x,0+)=(\partial_t u_-)(x,0+)=0, && (x,t)\in S^-,\\
&u_+(x,t)-u_-(x,t)=f(x,t), && (x,t)\in \Gamma.\\
&(T_+u_+)\ (x,t)-(T_-u_-)\ (x,t)=g(x,t), && (x,t)\in \Gamma\ldots
\end{aligned} \tag{2}$$

2. Function spaces

Let $m\in R$ and $p\in C$. We denote by $H_{m,p}(R^2)$ the space that coincides with $H_m(R^2)$ as a set but is equipped with the norm

$$\|u\|_{m,p}=\{\int_{R^2}(1+|p|^2+|\xi|^2)^m|\tilde{u}(\xi)|^2\,d\xi\}^{1/2},$$

where $\tilde{u}$ is the distributional Fourier transform of $u\in S'(R^2)$. If $S\subset R^2$, then $\overset{0}{H}_{m,p}(S)$ is the subspace of all $u\in H_{m,p}(R^2)$ such that $\operatorname{supp}u\subset\overline{S}$, and $H_{m,p}(S)$ is the space of the restrictions to S of all $v\in H_{m,p}(R^2)$. The norm on $H_{m,p}(S)$ is

$$\|u\|_{m,p;S}=\inf_{v\in H_{m,p}(R^2):v|_S=u}\|v\|_{m,p}.$$

$H_{-m,p}(R^2)$ is the dual of $H_{m,p}(R^2)$ with respect to the duality generated by the inner product $(\cdot,\cdot)_0$ in $L^2(R^2)$. The dual of $\overset{0}{H}_{m,p}(S)$ is $H_{-m,p}(S)$.

We denote by γ the trace operator that maps $H_{1,p}(S)$ continuously to the space $H_{1/2,p}(\partial S)$, which coincides as a set with $H_{1/2}(\partial S)$ but is equipped with the norm

$$\|f\|_{1/2,p;\partial S}=\inf_{u\in H_{1,p}(S):\gamma u=f}\|u\|_{1,p;S}.$$

We also consider $H_{-1/2,p}(\partial S)$, which is the dual of $H_{1/2,p}(\partial S)$ with respect to the duality generated by the inner product $(\cdot,\cdot)_{0;\partial S}$ in $L^2(\partial S)$.

For $\kappa > 0$ fixed, we introduce the complex half-plane $C_\kappa = \{p = \sigma + i\tau \in C : \sigma > \kappa\}$. Consider the space $H^L_{m,k,\kappa}(S)$ of all $\hat{u}(x,p), x \in S, p \in C_\kappa$, such that $U(p) = \hat{u}(\cdot, p)$ is a holomorphic mapping from C_κ to $H_m(S)$ and for which

$$\|\hat{u}\|^2_{m,k,\kappa;S} = \sup_{\sigma > \kappa} \int_{-\infty}^{\infty} (1 + |p|^2)^k \|U(p)\|^2_{m,p;S} \, d\tau < \infty.$$

The norm on $H^L_{m,k,\kappa}(S)$ is defined by this equality. The space $H^L_{\pm 1/2,k,\kappa}(\partial S)$ and its norm $\|\cdot\|_{\pm 1/2,k,\kappa;\partial S}$ are introduced similarly.

Finally, let $H^{L^{-1}}_{m,k,\kappa}(G)$ and $H^{L^{-1}}_{\pm 1/2,k,\kappa}(\Gamma)$ be the spaces of the inverse Laplace transforms u and f of all $\hat{u} \in H^L_{m,k,\kappa}(S)$ and $\hat{f} \in H^L_{\pm 1/2,k,\kappa}(\partial S)$, with norms

$$\|u\|_{m,k,\kappa;G} = \|\hat{u}\|_{m,k,\kappa;S}, \qquad \|f\|_{\pm 1/2,k,\kappa;\Gamma} = \|\hat{f}\|_{\pm 1/2,k,\kappa;\partial S}.$$

We denote by $\gamma^\pm$ the trace operators corresponding to $S^\pm$.

Making use of standard methods, we can easily obtain the variational formulation of the contact problem (2). To save space, we omit its explicit mention. A solution $\{u_+, u_-\} \in H^{L^{-1}}_{1,0,\kappa}(G^+) \times H^{L^{-1}}_{1,0,\kappa}(G^-)$ of the variational contact problem is called a weak solution of (2).

Theorem 1. For every $\kappa > 0$, $q \in H^{L^{-1}}_{-1,1,\kappa}(R^2 \times R_+)$, $f \in H^{L^{-1}}_{1/2,1,\kappa}(\Gamma)$ and $g \in H^{L^{-1}}_{-1/2,1,\kappa}(\Gamma)$, problem (2) has a unique weak solution $u = \{u_+, u_-\}$, where $u_\pm \in H^{L^{-1}}_{1,0,\kappa}(G^\pm)$. If $q \in H^{L^{-1}}_{-1,k,\kappa}(R^2 \times R_+)$, $f \in H^{L^{-1}}_{1/2,k,\kappa}(\Gamma)$, and $g \in H^{L^{-1}}_{-1/2,k,\kappa}(\Gamma)$, $k \in R$ then $u_\pm \in H^{L^{-1}}_{1,k-1,\kappa}(G^\pm)$ and

$$\|u_+\|_{1,k-1,\kappa;G^+} + \|u_-\|_{1,k-1,\kappa;G^-} \le c(\|q\|_{-1,k,\kappa;R^2 \times R_+} + \|f\|_{1/2,k,\kappa;\Gamma} + \|g\|_{-1/2,k,\kappa;\Gamma}).$$

3. Dynamic potentials and boundary integral equations

We define the single-layer potential

$$(V\alpha)(x,t)=\int_0^\infty\int_{\partial S}D(x-y,t-\tau)\alpha(y,\tau)ds_y d\tau,\qquad (x,t)\in R^2\times(0,\infty),$$

where $D(x,t)$ is a matrix of fundamental solutions for (1) vanishing when $t<0$. It is well known that $(V\alpha)(x,t)$ has the same boundary properties as the analogous potential in the static case. We denote by $(V_\pm\alpha_\pm)(x,t)$ the single-layer potentials constructed for the materials occupying the domains $S^\pm$, respectively. Representing the (weak) solution $\{u_+,u_-\}$ of (2) in the form

$$\begin{aligned} u_+(x,t)&=(V_+\alpha_+)(x,t), \quad (x,t)\in G^+ \\ u_-(x,t)&=(V_-\alpha_-)(x,t), \quad (x,t)\in G^- \end{aligned} \tag{3}$$

where α_+ and α_- are unknown densities defined on Γ and vanishing for $t<0$, we obtain the system of boundary integral equations

$$\begin{aligned} \gamma^+V_+\alpha_+-\gamma^-V_-\alpha_-&=f, \\ T_+V_+\alpha_+-T_-V_-\alpha_-&=g. \end{aligned} \tag{4}$$

Theorem 2. For any $\kappa>0$, and $k\in R$, $f\in H^{L^{-1}}_{1/2,k,\kappa}(\Gamma)$, $g\in H^{L^{-1}}_{-1/2,k,\kappa}(\Gamma)$, system (4) has a unique solution $\alpha_\pm\in H^{L^{-1}}_{-1/2,k-2,\kappa}(\Gamma)$. In this case, $\{u_+,u_-\}$ defined by (3) belongs to $H^{L^{-1}}_{1,k-1,\kappa}(G^+)\times H^{L^{-1}}_{1,k-1,\kappa}(G^-)$. If $k\geq 1$, then $\{u_+,u_-\}$ is the solution of problem (2).

We can easily introduce the dynamic analogue of the double-layer potential $(W\beta)(x,t)$ and represent the solution of (2) in terms of double-layer potentials. Moreover, it is possible to represent the solution in S^+ in the form of a single-layer potential and the solution in S^- in the form of a double-layer potential. All these representations lead to specific systems of boundary integral equations whose unique solvability has been proved. In all cases there hold statements analogous to those in Theorem 2.

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